

CARL LOUIS FERDINAND VON LINDEMANN

German analyst and geometer **Carl Louis Ferdinand von Lindemann** (April 12, 1852 – March 6, 1939) was the first to prove that π is transcendental, that is, it is not the root of any algebraic equation with rational coefficients. His methods were similar to those used by Charles Hermite in 1873 to prove that the constant base of the natural logarithms, e , is transcendental. Lindemann's achievement showed the impossibility of solving the ancient problem of constructing a square with the same area as a given circle using only a straightedge and a compass.



Lindemann was born in Hannover, the son of a modern language teacher at the Gymnasium, and his mother was the daughter of the headmaster of the Gymnasium. At the age of two, Ferdinand's father moved his family to Schwerin, where the young Lindemann attended school. He commenced his mathematical studies at Göttingen in 1870, where he was greatly influenced by Alfred Clebsch, the founder of a branch of algebraic geometry. Lindemann later published his notes from Clebsch's geometry lectures. As it was the practice of the time for German students to move from one university to another, Lindemann also attended the University at Munich and took his doctorate from Erlangen for a dissertation on non-Euclidean line geometry and its connection with non-Euclidean kinematics and statics, under the direction of Felix Klein.

Lindemann toured the great mathematical centers of Europe, visiting Oxford, Cambridge and London in England and spending time in Paris with Michel Chasles, Charles Hermite, and Camille Jordan. On returning to Germany, Lindemann was appointed professor of mathematics at the University of Königsberg. While there, both David Hilbert and Adolf Hurwitz joined the faculty. Hilbert had been

Lindemann's doctoral student at Königsberg. Lindemann accepted the chair of mathematics at the University of Munich in 1893, remaining there for the rest of his career.

Lindemann's proof that π is transcendental appeared in his 1882 paper "Über die Zahl π " (On the Number π). He first established that if, $x(1), x(2) \dots x(n)$ are distinct real or complex algebraic numbers and $p_1, p_2, \dots p_n$ are algebraic numbers, not all zero, then

$$p_1 e^{x(1)} + p_2 e^{x(2)} + \dots + p_n e^{x(n)} \neq 0$$

This is known as the Lindemann-Weierstrass theorem [also named for Karl Weierstrass] and the transcendence of e and π are corollaries of it. If e were algebraic, there would have to be rational numbers $p_1, p_2, \dots p_n$, not all zero, such that

$$p_1 e^0 + p_2 e^1 + \dots + p_n e^n = 0,$$

a contradiction of the Lindemann-Weierstrass Theorem. Thus e is transcendental. Now suppose that π is algebraic. Since it is known that the product of algebraic numbers is algebraic, it follows that $i\pi$ and $2i\pi$ are algebraic. Leonhard Euler established that

$$e^{ix} + 1 = 0$$

from which it follows that $e^{ix} = -1$ and $e^{2i\pi} = 1$, meaning

$$e^{ix} + e^{2i\pi} = -1 + 1 = 0$$

another contradiction of the Lindemann-Weierstrass Theorem. Thus, π is transcendental.

In 1761 Johann Heinrich Lambert proved that π was irrational but that was not sufficient to prove the impossibility of squaring the circle since many irrational numbers can be constructed using only a straightedge and compass. A geometrical construction is only possible with straightedge and compass if its equivalent algebraic form leads to equations with rational coefficients that can be solved by a process of successive extractions of square roots. For the quadrature of the circle to be possible with

Euclidean tools, the number π would have to be the solution of such an equation but as π is transcendental it is not a constructible number. That is, it is impossible to express π using only a finite number of integers, fractions, and their square roots. Thus it is impossible to construct, using straightedge and compass alone, a square whose area is equal to the area of a given circle.

The following verse seems apropos for this entry.

Pi goes on and on and on ...

And e is just as cursed.

I wonder: which is larger

When their digits are reversed?

- Anonymous

Telling some people that something is impossible is seen as a challenge. Even though it has long been known that the three construction problems of antiquity: the trisection of an angle, the squaring of a circle, and the duplication of a cube, are impossible using only a straightedge and compasses, there are a few individuals, who for some reason or other refuse to believe it. Many mathematicians receive unsolicited “solutions” of these three problems. In 1934, Alexander Gel'fond and Theodor Schneider independently gave an affirmative solution of David Hilbert's seventh problem. That is, they showed that if α and β are algebraic numbers, α is not 0 or 1, and β is irrational, then α^β is transcendental. Those looking for a real challenge could try to determine if $\pi + e$ or π^e is transcendental. At the present the status of both these numbers is unknown

Quotation of the Day: “What good is your beautiful investigation regarding π ? Why study such problems, since irrational numbers do not exist?” – Leopold Kronecker [commenting on Lindemann's

proof of the transcendence of π .]