G. F. B. Riemann

Georg Friedrich Bernhard Riemann (September 17, 1826 – July 20, 1866) was one of the leading and most influential 19th-century mathematicians. In his short life he introduced ideas of fundamental importance in partial differential equations, complex variable theory, differential geometry, and analytic number theory and laid the foundations for modern topology. He revolutionized the study of geometry even before he earned a paid university position. His work in differential geometry had a profound effect on modern theoretical physics and provided the mathematical basis for the general theory of relativity.

Riemann was born in Breselenz, Hanover (now Germany), second of six children of a Lutheran pastor of modest means. His father and a tutor named Schulz taught him at home until he was thirteen. In 1840 Riemann entered the Lyceum in Hannover while living with his grandmother. When she died, he moved to the Johanneum Gymnasium in Lüneburg. Because his mathematical interests were beyond the scope of the school, its director, Friedrich Schmalfuss, lent his own mathematical books to his precocious student. Six days after being lent Adrien Legendre's 859-page book *Théorie des nombres*, Riemann returned it announcing, "That was a wonderful book! I have mastered it!" And he had. Schmalfuss later wrote about his famous student, "His grasp for mathematical issues was immediately clear to me, and only a hint of a mathematical law would be enough for Riemann to see it realized with all its consequences in its simplest form."

In 1846 Riemann entered the University of Göttingen, where initially he studied theology and philology, but his mathematical interests remained strong, and his father gave him permission to study only mathematics. Despite the presence of Gauss, mathematical instruction at Göttingen was weak, prompting Riemann to move to the University of Berlin, where he attended the lectures of Carl Jacobi, Jacob Steiner, F.G. M. Eisenstein, and P.G.L. Dirichlet. Things improved at Göttingen when Wilhelm Weber returned to the University and so too did Riemann. He gained a strong foundation in theoretical physics from Weber and Johann Listing. Riemann completed his doctoral thesis “Foundations for a General Theory of Functions of one Complex Variable” in 1851. He introduced definitions of conformal mapping and simple connectivity, necessary for one of his main results, now known as the Riemann mapping theorem: “any simply connected domain of the complex plane having at least two boundary points can be conformally mapped onto the unit disk.” His treatment of the relations between varying complex numbers led to the idea of the Riemann surface, a multi-layered surface, on which a multivalued function of a complex variable can be interpreted as a single-valued function.

For the next two years Riemann worked on his *Habilitationsschrift* (probationary essay) and *Habilitationsvortrag* (probationary lecture), which would qualify him as a *Privatdozent.* A *docent* is a position at a European university that allows one to teach but without payment. A *Privatdozent* is an unsalaried lecturer paid only by his students’ fees and is not a member of the faculty. Students were required to prepare three lectures. Riemann expected Gauss to choose his lecture on Fourier series, “About the description of a function by arbitrary functions,” which was the subject of his essay. In this work, he gave the criteria for functions to be integrable, now spoken of as Riemann integrable. In addition, he obtained a sufficient condition for a Riemann integrable function to be represented by a Fourier series. Surprisingly, Gauss chose the last lecture listed by Riemann, which was the one for which he felt least properly prepared. Gauss claimed to have made the choice because he was curious how such a young man could handle the theme. Riemann’s 1854 thesis was titled “Über die Hypothesen welche der Geometrie zu Grunde liegen” (“On the Hypotheses That Lie at the Foundation of Geometry”), which is arguably the most remarkable *Habilitationsschrift* of all time.

In his lecture Riemann urged a revolutionary new way of looking at geometry. He believed that geometry should be considered as the study of manifolds of any number of dimensions in any kind of space. Geometry need not concern itself with points and lines or space in the ordinary sense but rather as a collection of ordered n-tuples, $(x_1, x_2, x_3, \ldots x_n)$ combined according to certain rules. Riemann ended the first part of the lecture by giving the definition of what today is called a Riemannian space. He also introduced the concept of the curvature of n-dimensional geometry and showed how to calculate this curvature at any point. Gauss was more than impressed with the results, praising the presentation with an enthusiasm rare for him when judging the work of others. Despite Gauss' appreciation for the depth of Riemann's ideas, the lecture was too far ahead of its time to be appreciated widely. It was not published until two years after Riemann's death and it was almost sixty years before it was fully understood. Riemann's development of the geometry of n-dimensional space became the basis of Einstein's general theory of relativity.

In his "Foundation of Geometry" Riemann rejected the belief that Euclid's axioms were self-evident truths about the space in which we lived. According to W.W. Rouse Ball in *A Short Account of the History of Mathematics,* "Riemann's hypothesis on geometry shows that as there are different kinds of line and surfaces, so there are different kinds of spaces of three dimensions." Riemann observed that experience does not suggest the infinite extent of a line, but only that it does not end. He distinguished between the unboundedness or endlessness of a line and infinite length. As an example he offered the notion of a circle. It is possible to move around the circle endlessly, yet its circumference is finite. In Riemannian geometry, the Euclidean axiom that a straight line extends indefinitely is replaced by the claim that it is merely unbounded.

Riemann also taught that experience does not suggest the existence of parallel lines, but rather that any two lines meet. Looking down a set of railroad tracks, it appears in the distance that the two rails at a constant distance from each other appear to meet. Euclid's Parallel Postulate is equivalent to the statement that "through a point not on a line, one and only one line passes through the point that is parallel to the line." Riemann replaced this with the assertion that "through a point not on a line, there are no lines passing through the point parallel to the line. A model of Riemannian geometry is a sphere with its great circles as bounded lines, each two of which will intersect not in one point but two. Riemann retained some of Euclid's axioms, and thus was able to arrive at some theorems identical to those of Euclid. However, the differences are what are most interesting. In Riemannian geometry the sum of the angles of a triangle are more than two right angles. The sum varies with respect to the area of the triangle and decreases towards 180° as the area of the triangle approaches zero. In Riemannian geometry all perpendiculars to a line meet in one point and similar triangles are congruent.

Riemann, who worked as Weber's assistant, was an outstanding and bold mathematician, but a timid, diffident man, who had a horror of speaking in public or attracting attention to himself. This proved a serious handicap to the shy young man's career aspirations. To overcome his problem, Riemann practiced meticulously every public utterance he was forced to make. The way his mind worked he thought of mathematics in jumps of intuition rather than rigorous development, which led to some criticism of his results by prominent mathematicians, including Karl Weierstrass who, although he found fault with Riemann's proofs, firmly believed in his results. One might have expected that the warm response of the Göttingen faculty and Gauss in particular to his *Habilitationsschrift* would have immediately earned Riemann a paid academic appointment but this was not to be. Despite his bitter disappointment, Riemann continued his mathematical investigations, focusing primarily on differential equations and Abelian functions.

When Gauss died in 1855, his chair went to Dirichlet. When Dirichlet died in 1859, Riemann succeeded him, at last escaping poverty. Shortly thereafter Riemann published another of his masterpieces, *The Theory of Abelian Functions*, in which he further developed the idea of Riemann surfaces and their topological properties. A few days later Riemann was elected to the Berlin Academy of Sciences. As was customary, new members had to report on their most recent research. In an incidental remark during his eight-page report *On the Number of Primes Less than a Given Magnitude,* Riemann made a guess, a hypothesis that changed the direction of mathematical research. He extended the zeta function to complex values, and offered the famous conjecture, now known by his name, that except for trivial exceptions the roots of his zeta function lie between 0 and 1 and that it seemed probable that these all have real part 1/2.

Following his guess, Riemann wrote, "One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation." Although the conjecture has been computationally tested and it has been found to be true for the first 200,000,001 zeros of the zeta function lie on the "critical line" $\mathcal{R}(s) = ½$ (the real part of the complex number *s*), the conjecture remains unresolved to this day – perhaps the most famous of all unsolved mathematical problems. Most mathematicians believe the conjecture is true and that its resolution holds the key to a variety of scientific and mathematical investigations, including the making and breaking of codes. There are many, many mathematical papers that contain some version of the line: "If the Riemann Conjecture is true, then the following is also true…."

In 1862, at the peak of his scientific career, Riemann married Elise Koch, a friend of his sister. In the first year of their marriage, Riemann, who never enjoyed good health and suffered from pleurisy, caught a heavy cold and was diagnosed with tuberculosis. The couple spent the year 1862-63 in Sicily and in 1864-1865 traveled through northern Italy, spending time with various Italian mathematicians including Enrico Betti. Riemann returned to Göttingen during the winter of 1865-66, but could only work a few hours each day. In June of 1866 he moved to the Italian village of Selasca on the shores of Lake Maggiore. Riemann was sustained by his deep religious faith, maintaining the main duty of the devout life to be: “Daily self-checking before the face of God.”

A month later, Riemann died peacefully at the age of not quite forty in a meadow while holding his wife and daughter in his arms. The number of Riemann’s published works is relatively small, but the extent to which it set directions for modern mathematics is almost immeasurable. In an article for the 1911 *Encyclopedia Britannica*, George Chrystal wrote: “Few as were the years of work allotted to him, and few are the printed pages covered by the record of his researches, his name is, and will remain, a household word among mathematicians. Most of his memoirs are masterpieces – full of original methods, profound ideas and far-reaching imagination.”

Quotation of the Day: “If only I had the theorems! Then I should find the proofs easily enough.” – G.F.B. Riemann