Moritz Pasch

If David Hilbert is the father of modern geometry, then **Moritz Pasch** (November 8, 1843 - September 20, 1930) is its grandfather, for he was first on the scene in raising most, and answering many, of the questions that arose following the advent of non-Euclidean geometry. Born in Breslau, Germany (now Wroclaw, Poland), Pasch attended the Elisabeth Gymnasium in Breslau before going on to the university, where he initially studied chemistry but changed to mathematics. After completing his dissertation, Pasch went to Berlin, where he studied with Weierstrass and Kronecker. Pasch taught at the University of Giessen from 1870 until he retired in 1911 to devote more time to his research. He led a simple life, saddened by the untimely deaths of his wife and two daughters. Pasch died while on a vacation trip away from Giessen.

The first seventeen years of his mathematical life Pasch worked in algebraic geometry and for the remaining 48 years in foundations of geometry, a field that didn't really exist before he took a hard look at Euclid's *Elements* and found a number of hidden assumptions in it that nobody had noticed before. For some two thousand years, Euclid's *Elements* was accepted by all as the one and only geometry and as the structural paradigm for any field of mathematics, with clearly stated axioms and definitions and all else rigorously proved from that starting point. Then, in the 19th century, Bolyai and Lobachevsky introduced the first non-Euclidean geometry, soon to be followed by Riemann's alternative non-Euclidean geometry. Perhaps the most important result of these inventions was the realization that geometries, including Euclid's, are human-made constructions that may bear some relation to the physical world but do not precisely represent it. With this realization mathematicians began to take note that Euclid's axioms needed some retooling and additions if they were to support the structure first proposed by him around 300 B.C. For example, Pasch noted that Euclid's "definitions" of his common notions, such as point, line and plane, were not really definitions at all. Stating that a point is "that which has no part" does not actually identify what a point is, for what exactly is meant by a "part"?

Pasch was not the first to recognize that any attempt to define *every* concept in terms of something more basic must result in an unending chain of definitions that would not really pin down anything. Even before Euclid, Aristotle saw this, and in more recent times George Peacock and George Boole maintained that some concepts must be *undefined*. Once these are identified, other concepts can be defined in terms of them. But it remained for Pasch to actually wrestle with this issue relative to Euclid's geometry. In modern geometry these primitive, undefined terms are usually taken to be "point," "line," and "plane." Pasch actually chose "point," "line segment"and"planar section" as his primitive terms. The last two were chosen rather than line and plane, because, as he pointed out, no one has actually had any experience of a line or a plane.

Pasch made his observations in his groundbreaking work, *Vorlesungen über neuere Geometrie* (1882). In it, he states that axioms are *not* "self-evident truths," as had been the conventional thinking for two millennia; rather, he asserted, axioms make assertions about undefined terms and these are the only assertions that can be used in developing the geometry. In a sense, undefined terms are given meaning implicitly by the axioms. Pasch insisted that while axioms may be suggested by experience, once they are chosen, all proofs of propositions must be based on them without further appeal of experience or physical reality. They are just assumptions designed to yield the theorems of a particular geometry.

Pasch was also the first to notice that in his proofs Euclid made use of properties of order without ever referring to them and certainly without introducing axioms about order. For instance, Euclid and everyone after him assumed if A, B, and C are three distinct points on a line then exactly one of them is between the other two. Similarly, Euclid, and all geometers before Pasch, had given proofs that depended on the property that a line separates a plane into two disjoint "half-planes." However, these assumptions about points, lines and planes went unstated, and if one was going to play the game of geometry according to the rules, they shouldn't be used in constructing proofs. Perhaps the reason no one had taken any notice of such oversights before is that the assumed properties are so "obvious," but that is just another way of saying everyone was making the same assumptions; it doesn't change the fact that they *are* assumptions. They may be very reasonable and useful assumptions, but they need to be identified; i.e., they should be stated as axioms.

Even though Euclid's *Elements* was an axiomatic means of presenting geometry, it had also been accepted as *the* description of physical space (i.e., "the truth"); no one had ever looked for such hidden assumptions. And no one had ever done a careful examination of what constitutes an axiom system. That examination became a major quest for those like Pasch who worked in the foundations of geometry. Not only did they have to divorce themselves from "reality" to identify the hidden assumptions of Euclid's (and others') geometry, but they also had to hammer out some agreement as to what features are necessary in order to have an axiom system. Pasch played a crucial role in the development of the axiomatic method, a central feature of modern mathematics.

The axiomatic method requires that *all* assumption (axioms) for a given system (Euclidean geometry, for instance) be stated, and that everyone constructing a proof in that system is restricted to using only those assumptions. People cannot simply introduce new assumptions whenever it suits their fancy, anymore than those playing baseball can change the rules in the middle of a game. It's not merely that this is "unsportsmanlike;" it can also be misleading. There is a famous "proof" that "all triangles are isosceles," which is "obviously" not true. Yet when the proof is presented it's quite convincing, and discovering the flaw – and of course, there must be a flaw – is not easy. The flaw arises from the diagram used to represent the information of the proposition that “all triangles are isosceles. The argument follows if certain betweeness of points occur, which happens when the diagram represents (or at least approximates) an isosceles triangle. There is a different betweeness property of these points for non-isosceles triangles. The bogus argument depends on the assumption that one of these betweeness relations holds rather than the other, an unstated assumption that the figure (not the axioms) fools one into accepting: but if one is not aware of the assumption made, then the “proof” seems correct.

Quotation of the Day: “… if geometry is to become a genuine deductive science, it is essential that the way in which inferences are made should be altogether independent of the *meaning* of the geometrical concepts, and also of the diagrams; all that need be considered are relationships between the geometrical concepts asserted by the propositions and definitions. In the course of deduction it is both advisable and useful to bear in mind the meaning of the geometrical concepts used, but this is *in no way essential*; in fact it is precisely when this becomes necessary that a gap occurs in the deduction and (when it is not possible to supply the deficiency by modifying the reasoning) we are forced to admit the inadequacy of the propositions invoked as the means of proof.” – Moritz Pasch, *Vorlesungen über neuere Geometrie*.