Felix Klein

In 1872, on the occasion of becoming a professor of mathematics at Erlangen, [**Christian**] **Felix Klein** (April 25, 1849 - June 22, 1925) delivered an inaugural address in which he described geometry as the study of properties of figures that remain invariant under a particular group of transformations. According to his scheme, plane Euclidean Geometry is the study of properties of figures including areas and lengths that are unchanged under the group of transformations made up of reflections, translations, and rotations. These so-called rigid transformations are equivalent to Euclid's unstated axiom that figures remain unchanged when moved about in a plane.

Klein's "Erlanger Programm," as it came to be known, gave the mathematical world a convenient means of classifying the various non-Euclidean geometries, according to properties left invariant under groups of transformations. Klein called the non-Euclidean geometry of Lobachevsky and Bolyai *hyperbolic geometry*; the geometry of Riemann, *elliptic geometry,* and *parabolic geometry* was reserved for Euclidean geometry. The characterization of the geometries is that in the Lobachevsky-Bolyai non-Euclidean geometry, for every line *l* and point *P*, not on the line, there are infinitely many distinct lines passing through *P* that are parallel to *l.* In the Riemann non-Euclidean geometry, there are no lines through *P* parallel to *l,* and in Euclidean geometry, there is exactly one line through *P* that is parallel to *l.*

Klein was born in Düsseldorf (then part of Prussia), where he attended the Gymnasium. After graduation, he studied mathematics at the University of Bonn and received his doctorate for a thesis, supervised by Julius Plücker, on line geometry and its applications to mechanics. Plücker died in 1868, leaving unfinished his work on a “new geometry” of a four-dimensional line space, in which a “figure” in the space is represented by a single equation $f(x, y, z, w) = 0$ with four coordinates for each point of the figure. Klein edited his teacher’s work and Plücker’s *New Geometry* was published that same year.

After serving as a medical orderly in the Franco-Prussian War, Klein was appointed professor at Erlangen, in 1872, where he developed his memorable synthesis of geometry. As important as this work was to mathematics, Klein felt his work on function theory was his major mathematical contribution. After leaving Erlangen, he spent five years at Munich’s *Technische Hochschule*, at which time he married Anne Hegel, the granddaughter of philosopher Georg Wilhelm Friedrich Hegel. From 1880 to 1886 Klein was at Leipzig, where he built a school that developed Riemann’s geometric approach to function theory. Klein’s quartic, also called Klein’s curve, is one of the most famous mathematical objects. Algebra students are familiar with the Klein four-group, the smallest non-cyclic group.

Klein’s plans were dealt a severe blow in 1882 when his health collapsed, followed by bouts of severe depression. Despite this, when he moved to Göttingen in 1886, where he stayed until his retirement in 1913, he was able to build a mathematical research group that served as a model for the world’s best mathematical centers. One of his most important coups was to induce David Hilbert to move to Göttingen in 1895. Klein also established *Mathematische Annalen* that specialized in complex analysis, algebraic geometry, invariant theory, and to a lesser extent, real analysis and group theory. He disapproved of the increasing tendency of mathematics to become highly abstract. He liked engineering applications. As he aged, Klein became insistent on keeping a careful schedule and having things done his way. He allotted time in his lectures for but two jokes, one for the fall semester and one for the spring semester. Even his daughter had to make an appointment to see him. The Göttingen students circulated the syllogism:

“There are two kinds of mathematicians at Göttingen – those who do what they want but not what Klein wants, and those who do what Klein wants but not what they want.

Klein is clearly of neither kind. Therefore Klein is not a mathematician."

Klein became interested in promoting general mathematical education at all levels. He championed modernizing mathematics instruction in Germany. Under his guidance as chairman, the German branch of the International Commission on Mathematical Instruction published many volumes on teaching mathematics. Klein's books on the history of mathematics, which in translation, can be appreciated by non-mathematicians are *Development of Mathematics in the 19th Century, Elementary Mathematics from an Advanced Standpoint, Famous Problems of Elementary Geometry,* and *The Mathematical Theory of the Top.*

It wouldn't do to leave this entry without mentioning the "Klein bottle," a one-sided closed surface with no boundaries. In topology, a Möbius Strip, named for August Ferdinand Möbius, is a one-sided surface, formed by taking a long rectangular strip of paper [Figure 4.10], and pasting the two ends together after giving it half a twist. Klein went Möbius one step further. He imagined sewing two Möbius strips together to create a single sided bottle with no boundary. Its inside is its outside. It contains itself. While anyone easily can construct a Möbius Strip [Figure 4.11], the construction of the Klein bottle is impossible in our universe, but if we were allowed to go outside it, the construction is possible in four-dimensional space. Various artists and artisans have created two-dimensional images and three-dimensional projections of the Klein bottle. Figure 4.12 shows a strip of paper and the identification of sides that would have to be made to obtain a Klein bottle [Figure 4.13].

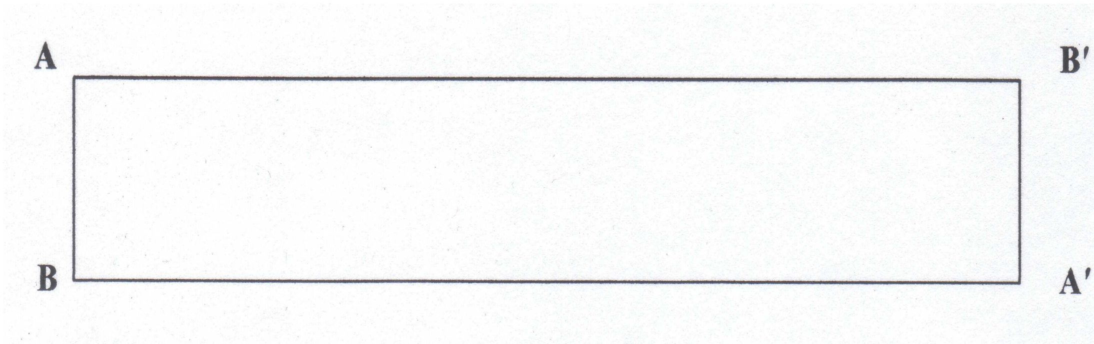
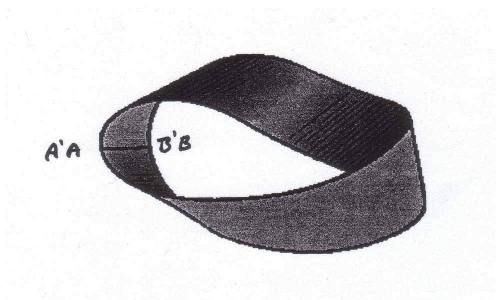


Figure 4.10



Mobius Strip

Figure 4.11

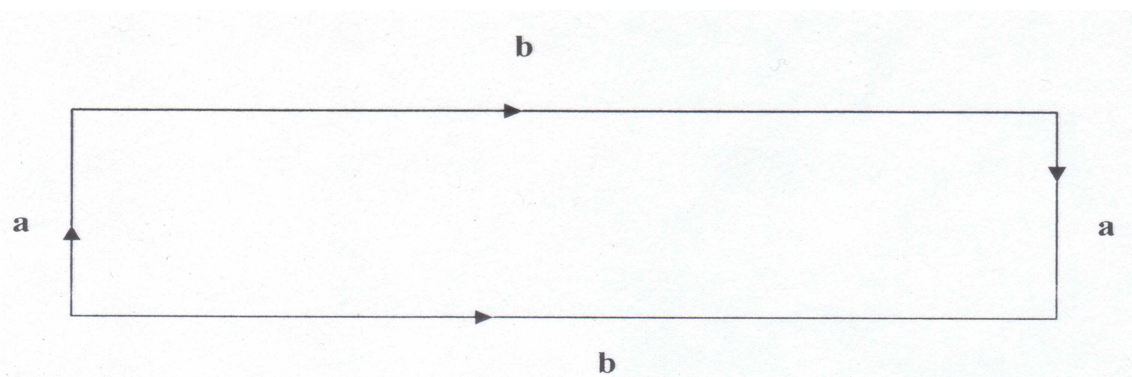


Figure 4. 12

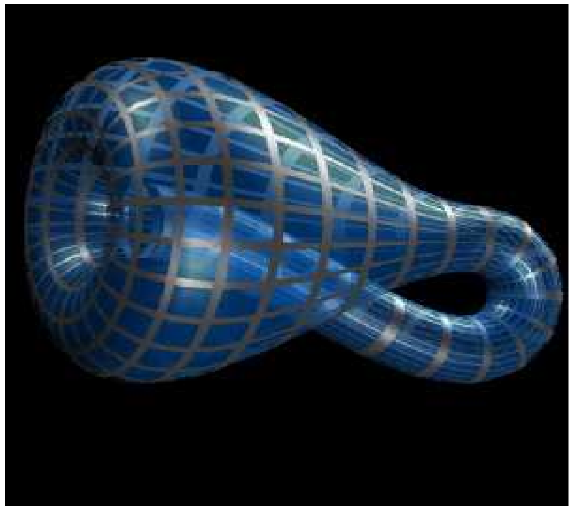


Figure 4.13

Manufacturers have found that certain machinery belts, which have been twisted into the shape of a Möbius strip, actually last longer. The fact that the Klein bottle can't be constructed is not as important as that it can be imagined. It is possible mathematically to describe "objects," even if they don't exist in our physical universe. Practice with such objects allows mathematicians and scientists to think deeply about higher dimensional spaces that need not have geometrical reality to be useful. Klein was a strong advocate of the use of models in mathematics education reform. In 1883, Victor Schlegel constructed accurate projection models of the six regular four-dimensional polytopes. A number of modern artists found that such possibilities freed them to interpret "reality" in new, and to some, strange ways. According to Rudolf Hoppe in "Abstract 6.2," *Paul Niggli Symposium in ETH* (August 1984), Schlegel's projections solved Pablo Picasso's problem of how to paint a cow showing all features at the same time. Using computer animations, the film industry is able to show bizarre and "unreal" things on the screen.

Quotation of the Day: "In fact, mathematics [has] grown like a tree, which does not start at its tiniest rootlets and grow merely upward, but rather sends its roots deeper and deeper at the same time and rate that its branches and leaves are spreading upward. … We see, then, that as regards the fundamental investigation in mathematics, there is no final ending, and therefore on the other hand, no first beginning." – Felix Klein