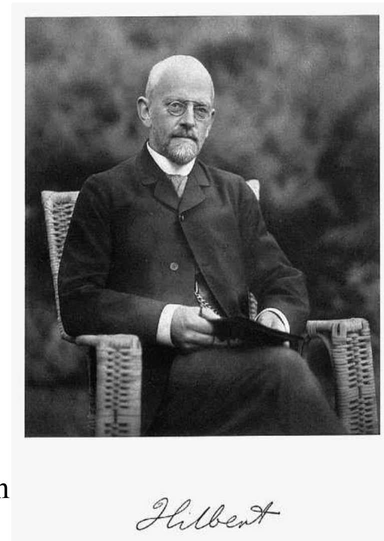
DAVID HILBERT

It is impossible in this brief space to give **David Hilbert** (January 23, 1862 – February 14, 1943) his due. For a large part of two centuries this mathematical giant dominated the world's mathematical development. Only Henri Poincaré of France could rival him at the time. Born in Königsberg, Prussia (now Kaliningrad, Russia), Hilbert attended the gymnasium and the university there. He received his doctorate in 1885 for a thesis on invariant theory, under the supervision of Carl Lindemann. Hilbert taught at Königsberg until 1895, when he was appointed to the chair of mathematics at the University of Göttingen, where he remained for the rest of his life. Because of his eminent position in the mathematical world after 1900, many universities tried to lure him away from Göttingen. In 1902, he was offered a chair at Berlin. Before turning it down he used it as leverage to get his friend Hermann Minkowski appointed to a chair at Göttingen.



Hilbert's mathematical achievements are far too many to do anything other than to hit the highlights, of which there were many. His work on invariant theory and algebraic number theory had enormous effect on both subjects. In 1888, Hilbert proved his famous Basis Theorem, that is, if every ideal in a ring R has a finite basis, so does every ideal in the polynomial ring $R[x]$. Hilbert had thus connected the theory of invariants to the fields of algebraic functions and algebraic varieties. When Felix Klein read the paper he wrote "I do not doubt that this is the most important work on general algebra that the [journal] has ever published." Beginning in 1893, Hilbert worked on a book on algebraic number theory. The result, *Zahlbericht*, published in 1897, was for half a century the source to go to for anyone who wanted to know anything about algebraic number theory.

In 1899, Hilbert proposed establishing a postulational basis for all mathematics and began with the foundations of geometry. It had long been observed that Euclid, knowingly or unknowingly made more assumptions in his *Elements* than he listed. But until Hilbert, no one had done much about it. After a systematic study of Euclid's theorems, Hilbert proposed and explained the significance of 21 axioms that would be used to form the basis of the subject. His results were contained in his *Grundlagen der Geometrie,* (The Foundations of Geometry) first published in 1899. It would go through nine editions over a period of more than sixty years, and is still consulted by mathematicians. Hilbert provided Euclidean geometry with a formal axiomatic setting that had the greatest influence on the subject since Euclid. Hilbert's work was a major influence in founding the axiomatic approach to mathematics, which has been one of the major characteristics of the field ever since.

In 1900, at the age of 38, David Hilbert presented a lecture "Mathematical Problems" before the Second International Congress of Mathematicians in Paris. In it, he proposed 23 problems, which crystallized mathematical thinking throughout the 20th century. In 1958, Paul Halmos wrote, "Mathematicians of this century have been significantly fortunate in having a ready-made and inspiring list of problems to work on. " The following is a list of the titles of the twenty-three problems.

1. Cantor's Problem of the Cardinal Number of the Continuum
2. The Compatibility of the Arithmetical Axioms
3. The Equality of the Volumes of Two Tetrahedra of Equal Bases and Equal Altitudes
4. Problem of the Straight Line as the Shortest Distance between Two Points
5. Lie's Concept of a Continuous Group of Transformations without the Assumption of the Differentiability of the Function Defining the Group
6. Mathematical Treatment of the Axioms of Physics
7. Irrationality and Transcendence of Certain Numbers
8. Problems of Prime Numbers

9. Proof of the Most General Law of Reciprocity in Any Number Field
10. Determination of the Solvability of a Diophantine Equation
11. Quadratic Forms with Any Algebraic Numerical Coefficients
12. Extension of Kronecker's Theorem on Abelian Fields to Any Algebraic Realm of Rationality
13. Impossibility of Solution of the General Equation of the 7th Degree by Means of Functions of Only Two Arguments
14. Proof of the Finiteness of Certain Complete Systems of Functions
15. Rigorous Foundations of Schubert's Enumerative Calculus
16. Problems of Topology of Algebraic Curves and Surfaces
17. Expressions of Definite Forms by Squares
18. Building up Space from Congruent Polyhedra
19. Are the Solutions of Regular Problems in the Calculus of Variations Always Necessarily Analytic?
20. The General Problem of Boundary Values
21. Proof of the Existence of Linear Differential Equations Having a Prescribed Monodromic Group.
22. Uniformization of Analytic Relations by Means of Automorphic Functions
23. Further Development of the Methods of the Calculus of Variations.

It was not merely Hilbert's reputation as the outstanding mathematician of his age that made these problems significant. They were the types of questions that mathematicians wanted to answer. Each problem in its own way is important enough and difficult enough to be of interest, and each led to the development of a large body of new mathematics. It isn't enough to announce the solutions to these problems – they must be peer reviewed before they can be accepted. Most of Hilbert's problems have been at least partially solved in the 20th century. Some were restated and new interpretations solved. Problem 10 has been solved negatively, i.e., it is impossible to derive the process that Hilbert sought for solving Diophantine equations. Problem 4 is considered too vague. The open questions are problems 8 and 16.

The Clay Mathematics Institute (CMI) is a private, non-profit foundation, based in Cambridge, Massachusetts. Dedicated to increasing and disseminating mathematical knowledge, to celebrate the new millennium, CMI has identified seven old and important mathematics questions that despite many attempts, mathematicians have failed to solve. CMI has allocated a $1 million prize for the solution of each problem. The seven "millennium" problems are:

1. P versus NP
2. The Hodge Conjecture
3. The Poincaré Conjecture
4. The Riemann Hypothesis
5. Yang-Mills Existence and Mass Gap
6. Navier-Stokes Existence and Smoothness
7. The Birch and Swinnerton-Dyer Conjecture

CMI has sweetened the pot for solving important, difficult, and far-reaching problems, echoing Hilbert's cry against ignorance: "We must know, we will know."

Among his many other contributions to mathematics and logic Hilbert published two volumes of *Grundlagen der Mathematik* (1934, 1939) that introduced his "proof theory," meant to be a direct check for the consistency of mathematics. In a 1931 paper, Kurt Gödel demonstrated with his incompleteness theorem that Hilbert's aim could not be realized. Towards the end of Hilbert's career, his work in integral equations led to the creation of the field of functional analysis. His work also established the basis for infinite dimensional space, later to be called Hilbert space. Few mathematicians of any century could match Hilbert's remarkable insight in mathematics.

As one might expect of a man who had so many complicated things on his mind, Hilbert displayed many of the stereotypical characteristics of an absent-minded and eccentric professor. Many of the stories that illustrate this are detailed by George Pólya in his article "Some Mathematicians I Have Known" published in the September 1969 issue of *The American Mathematical Monthly*. Hilbert only prepared his lectures in outline with the details to be worked out in the classroom, which actually isn't too bad an approach. In this way students can see how their instructor thinks and does mathematics. However, the flaw is that as happened often enough to Hilbert, the instructor gets stuck and cannot clear up a sticking point in a lecture. Hilbert was known for getting out of these difficulties by finally claiming the matter was trivial and moving on. Legend has it that in one class, Hilbert claimed that something was trivial, and then thought a bit more about the matter, and mumbled, "But is it trivial?" He then began pacing the classroom deep in thought. His pacing finally took him out of the room, leaving his students behind. Sometime later he burst back into the room, saying, "Yes, it is trivial."

Quotation of the Day: "It is difficult and often impossible to judge the value of a problem correctly in advance; for the final award depends upon the gain which science obtains from the problem. Nevertheless we can ask whether there are general criteria, which mark a good mathematical problem. An old French mathematician [Joseph-Louis Lagrange] said 'A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man you meet on the street.' This clarity and ease of comprehension, here insisted on for a mathematical theory, I should still demand for a mathematical problem if it is to be perfect; for what is clear and easily comprehended attracts, the complicated repels us." – David Hilbert