Gotthold Max Eisenstein

German mathematician [**Ferdinand**] **Gotthold Max Eisenstein** (April 16, 1823 - October 11, 1852) worked on a variety of topics, including the theory of forms and higher reciprocity laws, with the aim of generalizing Carl Friedrich Gauss's results in his *Disquistiones arithmeticae*. A polynomial is *homogeneous* if its terms are all of the same degree with respect to all the variables taken together. For instance, $x^2 - 3xy + y^2$ is homogeneous. A *form* is a homogeneous polynomial expression in two or more variables. In particular, a *bilinear form* is a polynomial of the second degree (*), which is homogenous of the first degree in variables $x_1, x_2, \ldots, x_n$ and in variables $y_1, y_2, \ldots, y_n$., with coefficients a_{ij} for $i, j = 1, 2, \ldots n$.

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j. \qquad (*)$$

For instance, the expression, $2x_1y_1 - 5x_2y_2 + 7x_3y_3$ is a bilinear form. There are trilinear forms, and multilinear forms of order m. Gauss made a major advance with a solution of the representation of numbers by binary quadratic forms. Eisenstein made the general extension from two to three indeterminates in his work *Neue Theoreme der höheren Arithmetik*.

As to reciprocity laws, if there is an integer x such that $x^2 - p$ is divisible by q, then p is said to be a quadratic residue of q. Otherwise p is said to be quadratic nonresidue of q. In 1808, Adrien-Marie Legendre introduced the symbol (p/q), defined as follows: For any number p and any prime q, $(p/q) = 1$ if p is a quadratic residue of q and $(p/q) = - 1$ if p is a quadratic nonresidue of q. It is understood that $(p/q) = 0$ if p divides q. The law of quadratic reciprocity states if p and q are distinct odd primes, then

$$(q/p)(p/q) = (-1)^{[(q-1)(p-1)]/4}$$

Gauss apparently first established the law when he was nineteen and gave the first rigorous demonstration of quadratic reciprocity in his *Disquistiones arithmeticae*. He later published four additional proofs. Gauss proceeded to search for reciprocity laws applicable to congruences of higher degree. He was able to state the law of cubic reciprocity and the law of biquadratic (quartic) reciprocity. Gauss did not publish his proof of the law of biquadratic reciprocity. Carl Jacobi gave a proof in his lectures at Königsberg in 1837, and Eisenstein published five proofs of the law, the first two appearing in 1844.

Eisenstein was born in Berlin, just after his family had converted from Judaism to the Protestant religion. All of his five brothers and sisters died of meningitis as children. Gotthold, who perhaps understandably was a hypochondriac all his life, also contracted the disease but survived. He wrote an autobiography in which he described his mother's major role in his early education and how by six he was able to understand proofs of mathematical theorems. While attending the Friedrich Wilhelm Gymnasium his mathematical talents were recognized and encouraged by his teachers, but he soon needed more instruction than they were able to give him, so he began to buy and study mathematics books on his own.

At seventeen, he attended lectures by P.G. Lejeune Dirichlet and other mathematicians at the University of Berlin. In 1842 Eisenstein bought a copy of Gauss' *Disquisitiones arithmeticae* and became fascinated with number theory. While in Ireland in 1843 he met Sir William Rowan Hamilton, who gave him a copy of his paper on Abel's work on the impossibility of solving quintic equations in general. Eisenstein was so stimulated he enrolled at the University of Berlin, and at the same time he submitted a paper to the Berlin Academy on cubic forms with two variables. The paper was refereed by August Crelle, who, as he did earlier with Niels Abel, once again showed his gift for recognizing talent by declaring that young Eisenstein was a potential genius. Eisenstein published 23 papers and two problems in *Crelle's Journal* in 1844.

At Berlin Eisenstein studied with Alexander von Humboldt, who arranged for poverty-stricken Eisenstein to obtain grants from the King, the Prussian government and the Berlin Academy. In 1844, Eisenstein spent two weeks at Göttingen visiting Gauss, who was extremely taken with the papers Eisenstein had sent him. Eisenstein's reputation, made for generalizing Gauss's results on quadratic forms and quadratic reciprocity, enabled Ernst Kummer to arrange for the University of Breslaw to award the twenty-two year old an honorary doctorate. In 1847 Eisenstein received his *habilitation* from the University of Berlin and began to lecture.

Despite remaining only a *Privatdozent*, and with his health continually deteriorating, Eisenstein published one treatise after another and received many honors. Early in 1852, sponsored by Dirichlet, he was elected to the Berlin Academy. On one occasion Eisenstein accessed his work in these words: "It is poor of achievements, deeds and merits, but perhaps it is not poor of good substance: it contains the resolutions and intentions for my future life and the germs of all good and all beauty that will unfold one day." Alas, it was not to be as he hoped. Humboldt was attempting to raise funds to send his protégé to Sicily to recover his health when Eisenstein died at age 29 of pulmonary tuberculosis.

Quotation of the Day: "There have been only three epoch-making mathematicians, Archimedes, Newton, and Eisenstein." – Attributed to Carl Friedrich Gauss. It is not known what criteria Gauss, not prone to flattery, used in assessing Eisenstein's contributions; nor whether the young man might have proved himself worthy of the designation had he been given the benefit of a long life. It is just possible that some important discoveries that could have been made by Eisenstein were lost, or at the very least delayed, by his premature death.